Exercise 64

The **left-hand** and **right-hand derivatives** of f at a are defined by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

and

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

(a) Find $f'_-(4)$ and $f'_+(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5-x} & \text{if } x \geq 4 \end{cases}$$

(b) Sketch the graph of f.

(c) Where is f discontinuous?

(d) Where is f not differentiable?

Solution

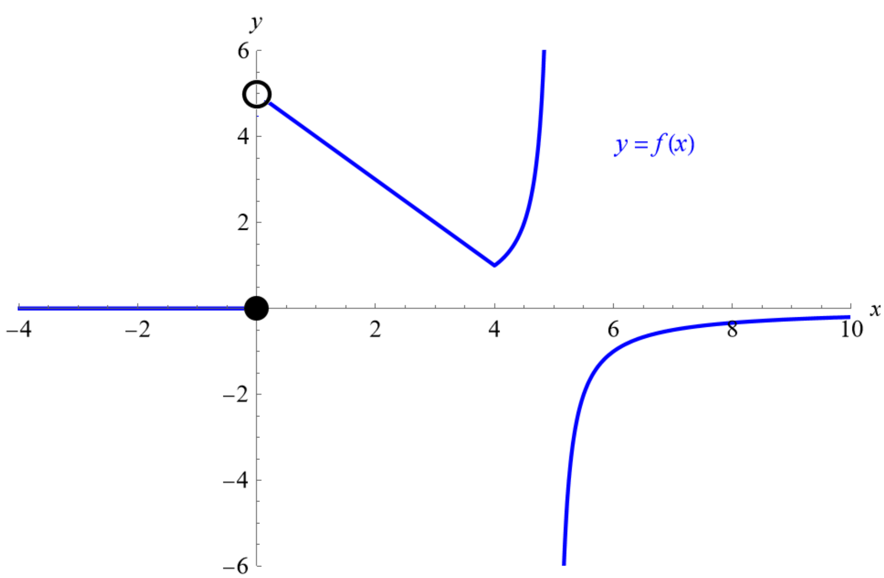
By the given definitions,

$$\begin{aligned} f'_-(4) &= \lim_{h \to 0^-} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h \to 0} \frac{[5 - (4+h)] - \dfrac{1}{5-(4)}}{h} \\ &= \lim_{h \to 0} \frac{(1-h) - 1}{h} \\ &= \lim_{h \to 0} \frac{-h}{h} \\ &= \lim_{h \to 0} (-1) \\ &= -1 \end{aligned}$$

and

$$
\begin{aligned}
f'_+(4) &= \lim_{h\to 0^+} \frac{f(4+h)-f(4)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{5-(4+h)} - \dfrac{1}{5-(4)}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{1-h} - 1}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{1-h} - \dfrac{1(1-h)}{1-h}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1-(1-h)}{1-h}}{h} \\
&= \lim_{h\to 0} \frac{1-(1-h)}{h(1-h)} \\
&= \lim_{h\to 0} \frac{h}{h(1-h)} \\
&= \lim_{h\to 0} \frac{1}{1-h} \\
&= \frac{1}{1} = 1.
\end{aligned}
$$

Since $f'_-(4) \neq f'_+(4)$, $f'(4)$ does not exist.



f is discontinuous at $x = 0$ and $x = 5$ and not differentiable at $x = 0$, $x = 4$, and $x = 5$.